# COMPUTATIONS OF MOMENTS FOR DISCOUNTED BROWNIAN ADDITIVE FUNCTIONALS

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## 1. Introduction

Let $(B_t)$ be the one dimensional standard Brownian motion and $(\ell_t^x)$ be its local time at $x$. Then the discounted local time at $x$ is defined by

$$L^x = \int_0^\infty e^{-s} d_s \ell_s^x. \tag{1.1}$$

And we also define the discounted time spent above $x$:

$$A^x = \int_0^\infty e^{-s} 1_{(B_s > x)} ds. \tag{1.2}$$

In [BW1], M. Baxter and D. Williams study the law of the functional $A = A^0$. In their approach, the following symmetry property is fundamental.

$$A \overset{law}{=} 1 - A \qquad under \quad P_0. \tag{1.3}$$

Moreover, with the help of the differentiability in $x$ of the Laplace transform of $A^x$, they obtained a double recurrence formula for the moments and its asymptotic law. In [BW2], they extended their considerations to a large class of diffusion processes.

In [Y1], the author studied the joint moments of $L(= L^0)$ and $A$, explaining the differentiability property obtained in [BW1] as a consequence of the following formulae:

$$A^x = \int_x^\infty dy \int_0^\infty e^{-s} d_s \ell_s^y = \int_x^\infty dy L^y \tag{1.4}$$

And the symmetry property may also be extended in the joint form:

$$(L, A) \overset{law}{=} (L, 1 - A) \qquad under \quad P_0. \tag{1.5}$$

Then, with the help of the right and left derivatives at $x = 0$ of the joint Laplace transform of $L^x$ and $A^x$, he obtained a double recurrence formula for joint moments.

Let $\mu(dx)$ be any Radon measure on $\mathbb{R}$ and define (the terminal value of) a general discounted Brownian additive functional:

$$F(\mu) = \int_{\mathbb{R}} \mu(dx) \int_0^\infty e^{-s} d_s \ell_s^x \tag{1.6}$$

as well as its moments function:

$$M_n(x, \mu) = E_x[F(\mu)^n]. \tag{1.7}$$

In this paper, we show a general recurrence formula for this sequence. Although our method is very simple, it may be applied to a general diffusion process. Here, we study only Brownian functionals. Using this, we can easily get the joint moments of $A$ and $L$. We also obtain some direct induction formulas. Moreover, we are also led to discuss certain integral equations, which are satisfied by the characteristic functions. We would like to mention some related papers which deal with the laws, or moments, of additive functionals of Markov processes, including Meyer[M], Pitman-Yor[PY1],[PY2].

## 2. A GENERAL FORMULA FOR MOMENTS

First, we define

$$L^x_{(k)} = \int_0^\infty e^{-ks} d_s \ell_s^x. \tag{2.1}$$

We omit the index $(k)$ when $k = 1$ and $x$ when $x = 0$. Thus we use the abbreviations $L = L^0_{(1)}, L^x = L^x_{(1)}$, and so on. Define the stopping time

$$T_a = \inf\{t; B_t = a\}. \tag{2.2}$$

We recall the following fundamental facts.

**Lemma 1.**

$$E_x[e^{-kT_y}] = e^{-\sqrt{2k}|x-y|}. \tag{2.3}$$

$$E_x[L^y_{(k)}] = \frac{1}{\sqrt{2k}} e^{-\sqrt{2k}|x-y|}. \tag{2.4}$$

*This latter quantity is the resolvent density of Brownian motion.*

*Proof.* The first formula is elementary. We show the second. Let $\theta$ be an independent exponential time with parameter $k$. It is well-known that $\ell_\theta^0$ is also an exponential variable with parameter $\sqrt{2k}$(see [Y2, Prop.3.2]). Therefore, we have

$$E_0[L^0_{(k)}] = E[\ell_\theta] = \frac{1}{\sqrt{2k}}. \tag{2.5}$$

By the Markov property, we get

$$E_x[L^0_{(k)}] = E_0[L^0_{(k)}]E_x[e^{-kT_0}] = \frac{1}{\sqrt{2k}}e^{-\sqrt{2k}|x|}. \tag{2.6}$$

□

We introduce the notation

$$F_{(k)}(t,\mu) = \int_{\mathbb{R}} \mu(dx) \int_t^\infty e^{-ks} d_s \ell_s^x, \tag{2.7}$$

and we write

$$F_{(k)}(\mu) = F_{(k)}(0,\mu), \tag{2.8}$$

which is a so-called discounted additive functional. In particular, we have

$$L^x_{(k)} = F_{(k)}(\delta_x(dy)). \tag{2.9}$$

We consider the moments (1.7). Since $\ell_t^x$ is jointly continuous and using the Markov property, we have

$$\begin{aligned}
M_n(x,\mu) &= -E_x[\int_0^\infty d\{F(t,\mu)^n\}] \\
&= nE_x[\int_{\mathbb{R}} \mu(dy) \int_0^\infty F(t,\mu)^{n-1}e^{-t}d_t\ell_t^y] \\
&= nE_x[\int_{\mathbb{R}} \mu(dy) \int_0^\infty M_{n-1}(B_t,\mu)e^{-nt}d_t\ell_t^y].
\end{aligned}$$

On the support set of $d_t\ell_t^y$, we have $B_t = y$. Therefore we get

$$\begin{aligned}
M_n(x,\mu) &= nE_x[\int_{\mathbb{R}} \mu(dy) \int_0^\infty M_{n-1}(y,\mu)e^{-nt}d_t\ell_t^y] \\
&= nE_x[\int_{\mathbb{R}} \mu(dy) M_{n-1}(y,\mu)L^y_{(n)}] \\
&= n\int_{\mathbb{R}} \mu(dy) M_{n-1}(y,\mu)E_x[L^y_{(n)}] \\
&= n\int_{\mathbb{R}} \mu(dy) M_{n-1}(y,\mu)\frac{e^{-\sqrt{2n}|x-y|}}{\sqrt{2n}} \qquad (\textit{by Lemma } 1).
\end{aligned}$$

We now introduce the integral operators:

$$H^{(n)}_\mu \phi(x) = \int_{\mathbb{R}} \mu(dy)\phi(y)\frac{e^{-\sqrt{2n}|x-y|}}{\sqrt{2n}}.$$

We suppress $\mu$ when the measure with respect to which it is related is obvious. Thus we proved

**Theorem 1.**

$$M_n(x,\mu) = nH^{(n)}M_{n-1}(x,\mu) = n!H^{(n)}H^{(n-1)}\cdots H^{(1)}1.$$

More generally, we have

**Theorem 2.** *Let $\mu(dx)$ and $\nu(dx)$ be any pair of positive Radon measures on $\mathbb{R}$. Then we have*

$$\begin{aligned} E_x[F(\mu)^n F(\nu)^m] &= n \int_{\mathbb{R}} \mu(dy) E_y[F(\mu)^{n-1} F(\nu)^m] E_x[L^y_{(n+m)}] \\ &\quad + m \int_{\mathbb{R}} \nu(dy) E_y[F(\mu)^n F(\nu)^{m-1}] E_x[L^y_{(n+m)}]. \end{aligned}$$

*Proof.* The proof is almost the same as above, because

$$\begin{aligned} E_x[F(\mu)^n F(\nu)^m] &= -E_x[\int_0^\infty d\{F(t,\mu)^n F(t,\nu)^m\}] \\ &= nE_x[\int_{\mathbb{R}} \mu(dy) \int_0^\infty F(t,\mu)^{n-1} F(t,\nu)^m e^{-t} d_t \ell^y_t] \\ &\quad + mE_x[\int_{\mathbb{R}} \nu(dy) \int_0^\infty F(t,\mu)^n F(t,\nu)^{m-1} e^{-t} d_t \ell^y_t]. \end{aligned}$$

□

**Remark 1.** *The above discussion clearly holds for a diffusion process, which has local times. In the most abstract sense, this is an application of the optional projection and the formula of integration by parts with respect to optional increasing processes (see* [DM]*). In a restricted sense, it can be understood in terms of Markov potential theory. Under some conditions, there exists a bijective correspondance between the AF $(A_t)$ and the associated measure $\nu_A$(so-called Revuz measure, see* [R]*):*

$$E_x[\int_0^\infty e^{-\alpha t} dA_t] = \int U^\alpha(x,y) \nu_A(dy).$$

*Then we can easily obtain*

$$E_x[F^n] = n \int U^n(x,y) E_y[F^{n-1}] \nu_A(dy),$$

*where*

$$F = \int_0^\infty e^{-t} dA_t.$$

**Example 1.** *(Cf.* [Y1, Prop. 1 ]*) We consider $L$ and its moments. Since $\mu(dy) = \delta_0(dy)$, we have*

$$H^{(n)}\phi(x) = \phi(0) E_x[L^0_{(n)}]. \tag{2.10}$$

*Therefore*

$$\begin{aligned} E[L^n] &= nE[L^0_{(n)}] E[L^{n-1}] \\ &= \sqrt{\frac{n}{2}} E[L^{n-1}] = \sqrt{\frac{n!}{2^n}}. \end{aligned}$$

**Example 2.** *Let $\mu(dy) = 1_{(y>0)}dy$. Then*

$$H^{(n)}\phi(x) = \int_0^\infty dy\phi(y)\frac{1}{\sqrt{2n}}e^{-\sqrt{2n}|x-y|}.$$

*Therefore, we have*

$$E_x[A^n] = \sqrt{\frac{n!}{2^n}}\int_0^\infty dy_1 e^{-\sqrt{2n}|x-y_1|}\int_0^\infty dy_2 e^{-\sqrt{2(n-1)}|y_2-y_1|} \cdots \int_0^\infty dy_n e^{-\sqrt{2}|y_n-y_{n-1}|}.$$

*And we can get for $x > 0$,*

$$E_x[A] = 1 - \frac{1}{2}e^{-\sqrt{2}x}$$

$$E_x[A^2] = 1 - e^{-\sqrt{2}x} + \frac{1}{2\sqrt{2}}e^{-2x}$$

$$E_x[A^3] = 1 - \frac{3}{2}e^{-\sqrt{2}x} + \frac{3}{2\sqrt{2}}e^{-2x} + e^{-\sqrt{6}x}(\frac{1}{4} - \frac{3}{4\sqrt{2}})$$

$$E_x[A^4] = 1-2e^{-\sqrt{2}x}+\frac{3\sqrt{2}}{2}e^{-2x}+e^{-\sqrt{6}x}(1-\frac{3}{\sqrt{2}})+e^{-\sqrt{8}x}(-\frac{1}{4}-\frac{\sqrt{3}}{4}+\frac{3\sqrt{6}}{8}).$$

*Clearly there exists a recurrence rule for this sequence, which will be established in the next section. Moreover, for $x < 0$, from the above or from the Markov property, we deduce the simple formula*

$$E_x[A^n] = e^{\sqrt{2n}x}E_0[A^n]. \tag{2.11}$$

**Example 3.** *Let $\mu(dy) = \delta_a(dy)$ and $\nu(dy) = \delta_b(dy)$. By Theorem 2, we have*

$$\begin{aligned}E_0[L^a(L^b)^n] &= E_a[(L^b)^n]E_0[L^a_{(n+1)}] + nE_b[L^a(L^b)^{n-1}]E_0[L^b_{(n+1)}] \\ &= e^{-\sqrt{2n}|b-a|}\sqrt{\frac{n!}{2^n}}\frac{e^{-\sqrt{2(n+1)}|a|}}{\sqrt{2(n+1)}} + nE_0[L^{a-b}(L)^{n-1}]\frac{e^{-\sqrt{2(n+1)}|b|}}{\sqrt{2(n+1)}}\end{aligned}$$

*Then we can get*

$$E_0[L^aL^b] = \frac{1}{2\sqrt{2}}e^{-\sqrt{2}|b-a|}(e^{-2|a|} + e^{-2|b|}),$$

$$E_0[L^a(L^b)^2] = \frac{1}{2\sqrt{3}}e^{-\sqrt{2}|b-a|}(e^{-\sqrt{6}|a|} + e^{-\sqrt{6}|b|}(1 + e^{-2|b-a|})).$$

## 3. MOMENTS OF L AND A

It will be quite convenient to consider the moments of linear combinations of $A$ and $L$, for which we introduce the notation:

$$M_n(x) = E_x[(\alpha A + \beta L)^n] \qquad M_n \equiv M_n(0) = E_0[(\alpha A + \beta L)^n] \tag{3.1}$$

$$M_n^- = E_0[(-\alpha A + \beta L)^n]$$

By Theorem 1, we have

$$M_n(x) = n \int \mu(dy)\, e_n(x-y) M_{n-1}(y), \tag{3.2}$$

where

$$\mu(dy) = \alpha 1_{(y>0)} dy + \beta \delta_0(dy), \tag{3.3}$$

$$e_n(y) = \frac{1}{\sqrt{2n}} e^{-\sqrt{2n}|y|}. \tag{3.4}$$

Thus, we get

$$M_n(x) = \alpha n \int_0^\infty dy\, e_n(x-y) M_{n-1}(y) + \beta \sqrt{\frac{n}{2}} M_{n-1} e^{-\sqrt{2n}|x|}. \tag{3.5}$$

By the symmetry of $A$, noting that $y > 0$, we obtain

$$\begin{aligned} M_{n-1}(y) &= E_y[(\alpha A + \beta L)^{n-1}] = E_y[(\alpha(1-A^-) + \beta L)^{n-1}] \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \alpha^{n-1-k} E_y[(-\alpha A^- + \beta L)^k] \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \alpha^{n-1-k} E_y[e^{-kT_0}] E_0[(-\alpha A^- + \beta L)^k] \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \alpha^{n-1-k} e^{-\sqrt{2k}y} M_k^- \end{aligned}$$

This formula shows that $M_n(y)$ has a simple form and coefficients of $e^{-\sqrt{2k}y}$ are associated to the moments. This relation is used in [Y1] or [BW1] as one of their recurrence formulae. By substituting this to (3.5), we obtain

$$M_n(x) = \alpha n \sum_{k=0}^{n-1} \binom{n-1}{k} \alpha^{n-1-k} M_k^- \int_0^\infty dy\, e_n(x-y) e^{-\sqrt{2k}y} + \beta \sqrt{\frac{n}{2}} M_{n-1} e^{-\sqrt{2n}|x|}. \tag{3.6}$$

Thus we have

**Proposition 1.**

$$M_n(x) = \frac{\sqrt{n}}{2}\tilde{M}_n(x) + \beta\sqrt{\frac{n}{2}}M_{n-1}e^{-\sqrt{2n}|x|}, \qquad (3.7)$$

*where*

$$\tilde{M}_n(x) = \begin{cases} \sum_{k=0}^{n-1}\binom{n-1}{k}\alpha^{n-k}M_k^-\left(\frac{e^{-\sqrt{2k}x}}{\sqrt{n}+\sqrt{k}} + \frac{e^{-\sqrt{2k}x} - e^{-\sqrt{2n}x}}{\sqrt{n}-\sqrt{k}}\right) & (x > 0) \\ \sum_{k=0}^{n-1}\binom{n-1}{k}\alpha^{n-k}M_k^-\frac{e^{\sqrt{2n}x}}{\sqrt{n}+\sqrt{k}} \quad (x < 0). & \end{cases} \qquad (3.8)$$

*Moreover, the function $\tilde{M}_n$ is $C^1$, and we also have the direct induction formula on the moments:*

$$M_n = \frac{\sqrt{n}}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}\alpha^{n-k}\frac{M_k^-}{\sqrt{n}+\sqrt{k}} + \beta\sqrt{\frac{n}{2}}M_{n-1}. \qquad (3.9)$$

*Proof.* It suffices to prove the differentiability of $\tilde{M}_n$ at 0, which follows from:

$$\frac{-\sqrt{2k}}{\sqrt{n}+\sqrt{k}} + \frac{-\sqrt{2k}+\sqrt{2n}}{\sqrt{n}-\sqrt{k}} - \frac{\sqrt{2n}}{\sqrt{n}+\sqrt{k}} = \sqrt{2}\left(\frac{-(\sqrt{n}+\sqrt{k})}{\sqrt{n}+\sqrt{k}} + \frac{\sqrt{n}-\sqrt{k}}{\sqrt{n}-\sqrt{k}}\right) = 0.$$

□

Define

$$\Phi(\alpha,\beta;x) = E_x[e^{\alpha A + \beta L}]. \qquad (3.10)$$

Then we have

**Proposition 2.** *The joint Laplace transform of $(A, L)$ is given by:*

$$\Phi(\alpha,\beta;x) = 1 + (e^\alpha - 1)\sum_{n=0}^{\infty}\frac{e^{-\sqrt{2n}x}M_n^-}{n!} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{M_n^-}{n!}\sum_{k=1}^{\infty}\frac{\alpha^k e^{-\sqrt{2(n+k)}x}}{k!}\left(1+\sqrt{\frac{n}{n+k}}\right)$$
$$+ \frac{\beta}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{e^{-\sqrt{2(n+1)}x}M_n}{n!\sqrt{n+1}} \qquad (x > 0) \quad (3.11)$$

$$\Phi(\alpha,\beta;x) = 1 + \frac{1}{2}\sum_{n=0}^{\infty}\frac{M_n^-}{n!}\sum_{k=1}^{\infty}\frac{\alpha^k e^{\sqrt{2(n+k)}x}}{k!}\left(1 - \sqrt{\frac{n}{n+k}}\right)$$
$$+ \frac{\beta}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{e^{\sqrt{2(n+1)}x}M_n}{n!\sqrt{n+1}} \qquad (x < 0) \quad (3.12)$$

**Corollary 1.**

$$\frac{dM_n}{dx}(+0) - \frac{dM_n}{dx}(-0) = -2\beta n M_{n-1}. \tag{3.13}$$

$$\frac{d\Phi}{dx}(+0) - \frac{d\Phi}{dx}(-0) = -2\beta\Phi(0). \tag{3.14}$$

*Proof.* The first relation is obtained from Proposition 1 and the second is immediately deduced from the first. □

Moreover, we can obtain a general relation on the differentiability of moment functions. We use the notation in the previous section.

**Proposition 3.** *Suppose that $M_n(x) = M_n(x, \mu)$ is finite. Then we have*

$$\frac{dM_n}{dx}(a+) - \frac{dM_n}{dx}(a-) = -2n\mu(\{a\})M_{n-1}(a). \tag{3.15}$$

*Generally, let $f(x)$ be a $C^1$-function and suppose that $\Phi(x) = E_x[f(F(\mu))]$ is finite. Then we have*

$$\frac{d\Phi}{dx}(a+) - \frac{d\Phi}{dx}(a-) = -2\mu(\{a\})E_a[f'(F(\mu))]. \tag{3.16}$$

*Proof.* We decompose the measure

$$\mu(dy) = \mu(dy)1_{\{y\neq a\}} + \mu(\{a\})\delta_a(dy) \equiv \mu_1 + \mu_2.$$

Then, by Theorem 1, we have

$$\begin{aligned} M_n(x) &= n\int \mu_1(dy)\, e_n(x-y)M_{n-1}(y) + n\int \mu_2(dy)\, e_n(x-y)M_{n-1}(y) \\ &= n\int \mu_1(dy)\, e_n(x-y)M_{n-1}(y) + n\,\mu(\{a\})e_n(x-a)M_{n-1}(a), \end{aligned}$$

where $e_n$ is defined by (3.4). For $|h| \le \epsilon$, it is easy to see

$$\left|\frac{e_n(x+h-y) - e_n(x-y)}{h}\right| \le \sqrt{2n}\, e^{\epsilon} e_n(x-y), \tag{3.17}$$

by the Lebesgue dominated convergence theorem, we get the differentiability of the first term. And then, the second term gives the first assertion. The second assertion is also proved similarly. □

**Remark 2.** *(a)The relation* (3.16) *is found in* [Y1, formula (10)] *and is used there to get the joint moments. Moreover, a further extension of this relation is also given there.*

*(b) Suppose that $\mu(dy)$ is of the form $\eta(y)dy$ for a continuous function $\eta$ in a neighbourhood of $x$. Then $E_x[f(F)]$ is twice differentiable near $x$ and we can obtain*

$$(E_x[f(F)])'' = 2E_x[Ff'(F)] - 2\eta(x)E_x[f'(F)].$$

*We omit the proof. The complete discussion will appear elsewhere.*

## 4. Integral equations

We define an integral operator by

$$K\phi(\alpha) = \frac{1}{\sqrt{\pi}}\int_0^1 ds(\log\frac{1}{s})^{-\frac{1}{2}}\phi(s\alpha). \tag{4.1}$$

We denote the differential operator by $D(=\dfrac{d}{d\alpha})$.

**Lemma 2.** *The following properties hold:*

(i) $K1 = 1$.

(ii) $K[\alpha^n] = \dfrac{\alpha^n}{\sqrt{n+1}}$.

(iii) $K^2\phi = \dfrac{1}{\alpha}\displaystyle\int_0^\alpha \phi(t)dt(= H\phi)$,
*where $H$ is the so-called Hardy operator.*

(iv) $K\phi = 0 \Rightarrow \phi = 0$.

(v) $\alpha K^2 D\phi = \phi - \phi(0)$.

(vi) $KD\alpha K\phi = D\alpha K^2\phi = \phi$.

*Proof.* The proofs are almost direct and elementary. We only show (iv)and (vi). We look at (iv) first. If $K\phi = 0$, then $K^2\phi = 0$. By (iii), we conclude (iv). To see (vi), we set $u = KD\alpha K\phi$. Then we have

$$\alpha K u = \alpha K^2 D(\alpha K\phi) = \alpha K\phi \qquad \text{(by (v))}. \tag{4.2}$$

By (iv), we obtain $u = \phi$. The equality $D\alpha K^2\phi = \phi$ is direct by (iii). ☐

**Remark 3.** *The property (iii) is interesting. More generally, we can define the family of integral operators:*

$$K^{(p)}\phi(\alpha) = \frac{1}{\Gamma(p)}\int_0^1 ds(\log\frac{1}{s})^{p-1}\phi(s\alpha), \tag{4.3}$$

*where $p$ is a positive parameter. Then it is easy to see*

$$K^{(p)}[\alpha^n] = \frac{\alpha^n}{(n+1)^p}. \tag{4.4}$$

*Therefore we have*

$$K^{(p)}K^{(q)} = K^{(p+q)}. \tag{4.5}$$

*In particular, we have:*

$$K^{(1/2)}K^{(1/2)} = K^{(1)} = H. \tag{4.6}$$

Now we deduce the integral equations. We note the integration

$$\frac{1}{\sqrt{\pi}}\int_0^1 ds(\log\frac{1}{s})^{-\frac{1}{2}}e^{s\alpha}s^k = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!\sqrt{n+k+1}} \qquad (by \quad Lemma\ 2\ (ii)). \tag{4.7}$$

In Proposition 2, setting $x = 0$, we have

$$\begin{aligned}\Phi(\alpha,\beta) &= 1 + \frac{1}{2}(e^{\alpha}-1)E_0(e^{F^-}) - \frac{1}{2}\sum_{k=1}^{\infty}\frac{M_k^-\sqrt{k}}{k!}\{\frac{1}{\sqrt{\pi}}\int_0^1 ds(\log\frac{1}{s})^{-\frac{1}{2}}e^{s\alpha}s^{k-1} - \frac{1}{\sqrt{k}}\}\\ &\quad + \frac{\beta}{\sqrt{2\pi}}\int_0^1 ds(\log\frac{1}{s})^{-\frac{1}{2}}E_0(e^{sF})\\ &= 1 + \frac{1}{2}(e^{\alpha}-1)E_0(e^{F^-}) - \frac{1}{2\pi}\int_0^1 ds(\log\frac{1}{s})^{-\frac{1}{2}}e^{s\alpha}\int_0^1 dt(\log\frac{1}{t})^{-\frac{1}{2}}E_0[F^-e^{stF^-}]\\ &\quad + \frac{1}{2}(E_0(e^{F^-})-1) + \frac{\beta}{\sqrt{2\pi}}\int_0^1 ds(\log\frac{1}{s})^{-\frac{1}{2}}E_0(e^{sF})\end{aligned}$$

where

$$F = \alpha A + \beta L \qquad and \quad F^- = -\alpha A + \beta L.$$

On the other hand, we have

$$\alpha + F^- \stackrel{law}{=} F. \tag{4.8}$$

Therefore we can obtain

$$\begin{aligned}\Phi(\alpha,\beta) &= 1 + \frac{\alpha}{\pi}\int_0^1\int_0^1 ds\,dt(\log\frac{1}{s}\log\frac{1}{t})^{-\frac{1}{2}}e^{s\alpha-st\alpha}E_0[e^{stF}]\\ &\quad - \frac{1}{\pi}\int_0^1\int_0^1 ds\,dt(\log\frac{1}{s}\log\frac{1}{t})^{-\frac{1}{2}}e^{s\alpha-st\alpha}E_0[Fe^{stF}]\\ &\quad + \beta\sqrt{\frac{2}{\pi}}\int_0^1 ds(\log\frac{1}{s})^{-\frac{1}{2}}E_0(e^{sF})\end{aligned}$$

We define

$$\psi(\beta) = E_0[e^{\beta L}], \qquad \phi(\alpha) = E_0[e^{\alpha A}], \qquad \Phi(\alpha) = E_0[e^{\alpha(A+\nu L)}].$$

Then by the above formula, we have the following integral equations

$$\psi(\beta) = 1 + \frac{1}{\sqrt{2}}\beta K\psi(\beta) \tag{4.9}$$

$$\phi(\alpha) = 1 + \alpha K(e^{\alpha}K(e^{-\alpha}(\phi - \phi'))) \tag{4.10}$$

$$\Phi(\alpha) = 1 + \alpha K(e^{\alpha} K(e^{-\alpha}(\Phi - \Phi'))) + \alpha\nu\sqrt{2}K\Phi. \tag{4.11}$$

By Example 1 in section 2, we know

$$\psi(\beta) = \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!2^n}}, \tag{4.12}$$

which agrees with (4.9).

**Proposition 4.** *Under the condition* $\phi(0) = 1$ *and* $\Phi(0) = 1$*, the following are respectively equivalent to* (4.10)*and* (4.11)*:*

$$e^{-\alpha} K\phi' = -K((e^{-\alpha}\phi)'), \tag{4.13}$$

$$e^{-\alpha} K\Phi' = -K((e^{-\alpha}\Phi)') + \nu\sqrt{2}e^{-\alpha}\Phi. \tag{4.14}$$

*Proof.* We only show the first equation, since the second can be shown in the same way. By Lemma 2(v), we can write (4.10) as

$$\alpha K^2 D\phi = \alpha K(e^{\alpha} K(e^{-\alpha}(\phi - \phi'))).$$

Thus we get

$$KD\phi = e^{\alpha} K(e^{-\alpha}(\phi - \phi'))) = -e^{\alpha} K(D(e^{-\alpha}\phi)).$$

□

These equations have a unique solution in the space of analytic functions. In fact, by expanding these with respect to the parameter, we again get the direct induction formulas for the moments. For example, from (4.14), we get

$$M_n = \frac{1}{2} \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-1-k} (1 + \frac{\sqrt{k} + \sqrt{2}(n-k)\nu}{\sqrt{n}}) M_k, \tag{4.15}$$

where $M_n = E[(A + \nu L)^n]$.

We note that the equation (4.14) is an extension of the Baxter and Williams characterization of $A$. Define

$$h(\alpha) = \alpha K(e^{\alpha}) = \frac{\alpha}{\sqrt{\pi}} \int_0^1 ds (\log \frac{1}{s})^{-\frac{1}{2}} e^{s\alpha}. \tag{4.16}$$

Using the symmetry property, we see that (4.14) can be written

$$e^{-\alpha} E[h(\alpha A + \beta L)] = -E[h(-\alpha A + \beta L)] + \sqrt{2}\beta e^{-\alpha} E[e^{\alpha A + \beta L}], \tag{4.17}$$

where $\beta = \alpha\nu$. When $\beta$ is equal to zero, this is just the formula given in [BW1].

Finally, we point out that the symmetry property of $A$ is implied by (4.14), since this is invariant under the transformation:

$$e^{-\alpha}\Phi \to \Phi, \alpha \to -\alpha.$$

## References


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